ELLIPTIC K3 SURFACES ADMITTING A SHIODA-INOSE STRUCTURE

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1. Introduction

An automorphism of a K3 surface X is called symplectic if it acts on $H^{2,0}(X)$ trivially. Such automorphisms were studied by Nikulin in [N1]. He proved that a symplectic involution ι has eight fixed points and the minimal resolution $Y \to X/\langle\iota\rangle$ of eight nodes is again a K3 surface. In [SI], Shioda and Inose proved that every K3 surface X with maximal Picard number 20 has a symplectic involution ι such that Y is a Kummer surface, and that the rational quotient map $\pi : X \dashrightarrow Y$ induces a Hodge isometry $T_X(2) \cong T_Y$, where T_X is the transcendental lattice of X. In general, we say that a K3 surface X admits a Shioda-Inose structure if X has such an involution. This definition is due to Morrison ([Mo]), and he proved that a K3 surface X admits a Shioda-Inose structure if and only if there exists an Abelian surface A and a Hodge isometry $T_X \cong T_A$. Since the transcendental lattice of a general $(1,d)$-polarized Abelian surface is $M_d = \mathrm{U} \oplus \mathrm{U} \oplus \langle -2d\rangle$, a K3 surface X with Picard number 17 admits a Shioda-Inose structure if and only if $T_X \cong M_d$, namely $NS(X) \cong \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \langle 2d\rangle$. However, to the best of author's knowledge, an explicit example of a 3-dimensional family of such K3 surfaces with the involution ι is known only for $d = 1$ (Appendix in [GL], [K]) and for $d = 2$ ([vGS]). In [K] and [vGS], K3 surfaces X are given as elliptic surfaces with a 2-torsion section σ, and ι is given by the fiberwise translation by σ. In this situation, the rational quotient map $X \dashrightarrow Y$ is just an isogeny of degree 2 between elliptic curves over $\mathbb{C}(t)$, and we have a rational map $Y \dashrightarrow X$ of degree 2 as the dual isogeny. This gives a geometric realization of Kummer sandwich theorem $Y \dashrightarrow X \dashrightarrow Y$ which was proved by Ma ([Ma]).

In this short note, we show that such pairs of elliptic K3 surfaces exist only for $d = 1, 2, 3, 5, 7$ under the hypothesis that the Mordell-Weil rank is 0 (Theorem 2.6), and we construct X and Y explicitly for these values of d.

2. Elliptic K3 surfaces with a 2-torsion

2.1. Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface with the zero section o. If X has a 2-torsion section σ, it is given by the Weierstrass equation

$$y^2 = x(x^2 + a(t)x + b(t)), \qquad \deg a(t) \le 4, \quad \deg b(t) \le 8$$

with the projection $f(x,y,t) = t \in \mathbb{P}^1$, and $\sigma = \{x = y = 0\}$. Let ι be the translation by σ. It is a Nikulin involution, and we have a K3 surface Y by resolving eight nodes on $X/\langle\iota\rangle$. The rational quotient map $\phi : X \dashrightarrow Y$ is regarded as an isogeny between elliptic curves over $K = \mathbb{C}(t)$ with the kernel $\{o,\ \sigma\}$. The Weierstrass model of Y is

$$Y : \quad y^2 = x(x^2 - 2a(t)x + a(t)^2 - 4b(t)),$$

and the isogeny ϕ and the dual isogeny $\hat{\phi}$ is given by

$$\phi : X \longrightarrow Y, \quad (x,y) \mapsto (\frac{y^2}{x^2}, \frac{y(x^2 - b(t))}{x^2}),$$

$$\hat{\phi} : Y \longrightarrow X, \quad (x,y) \mapsto (\frac{y^2}{4x^2}, \frac{y(x^2 - a(t)^2 + 4b(t))}{8x^2})$$

([ST], Chapter III. 4). We denote the projection $(x,y,t) \mapsto t$ by $g : Y \to \mathbb{P}^1$. Up to constants, the discriminants of X and Y are

$$\Delta_X(t) = b^2(a^2 - 4b), \qquad \Delta_Y(t) = b(a^2 - 4b)^2.$$

For general $a(t)$ and $b(t)$, singular fibers of X and Y are $8I_1 + 8I_2$ and Mordell-Weil groups are $X(K) \cong Y(K) \cong \mathbb{Z}/2\mathbb{Z}$. Transcendental lattices and Néron-Severi groups are

$$T_X \cong T_Y \cong \mathrm{U} \oplus \mathrm{U} \oplus N, \quad NS(X) \cong NS(Y) \cong \mathrm{U} \oplus N$$

where N is the Nikulin lattice ([vGS]).

2.2. We are interested in $a(t)$ and $b(t)$ such that the transcendental lattice T_X of the corresponding K3 surface X is $M_d = \mathrm{U} \oplus \mathrm{U} \oplus \langle -2d \rangle$. To find such $a(t)$ and $b(t)$, let us study configurations of possible singular fibers. For our purpose, Shimada's list ([Shim] and [BK]) is useful, but here we make arguments self-contained as possible. We denote the simple points of a singular fiber $f^{-1}(\nu)$ by $f^{-1}(\nu)^\sharp$, which has a natural group structure. Since the specialization map $X(K)_{tor} \to f^{-1}(\nu)^\sharp_{tor}$ on the torsion subgroup is injective ([Mi], Corollary VII.3.3) and $\sigma \in X(K)$ is of order two, X admits singular fibers of Kodaira's type I_n, I_n^*, III and III^*. Fundamental invariants for these fibers are summarized in the following table, where L_ν is the (negative definite) Dynkin lattice generated by components which do not intersect with o, m_ν is the number of components, $m_\nu^{(1)}$ is the number of simple components, n_ν is the number of fixed points by ι and $c(t) = a(t)^2 - 4b(t)$.

$f^{-1}(\nu)$	I_n		I_{2k}^*		I_{2k+1}^*	III	III^*
$f^{-1}(\nu)^\sharp$	$\mathbb{C}^* \times (\mathbb{Z}/n\mathbb{Z})$		$\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2$		$\mathbb{C} \times (\mathbb{Z}/4\mathbb{Z})$	$\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})$	
L_ν	A_{n-1}		D_{2k+4}		D_{2k+5}	A_1	E_7
$\mathrm{ord}_\nu \Delta_X(t)$	n		$2k+6$		$2k+7$	3	9
$m_\nu(X)$	n		$2k+5$		$2k+6$	2	8
$m_\nu^{(1)}(X)$	n		4		4	2	2
ι	(i)	(ii)	(i)	(ii)	(i)	-	-
$n_\nu(X)$	n	0	$2k+2$	2	$2k+3$	1	3
$g^{-1}(\nu)$	I_{2n}	$I_{n/2}$	I_{4k}^*	I_k^*	I_{4k+2}^*	III	III^*
$\mathrm{ord}_\nu \Delta_Y(t)$	$2n$	$n/2$	$4k+6$	$k+6$	$4k+8$	3	9
$\mathrm{ord}_\nu b(t)$	0	$n/2$	2	$k+2$	2	1	3
$\mathrm{ord}_\nu c(t)$	n	0	$2k+2$	2	$2k+3$	1	3

These are very well known(see e.g. [Mi], [SS], [T]), except perhaps n_ν and the type of the fiber $g^{-1}(\nu)$ (Last three columns are determined from $\mathrm{ord}_\nu \Delta_X(t)$ and the fiber type of $g^{-1}(\nu)$). Here we explain the action of ι on I_n and I_n^*. First of all, note that an involution on $\mathbb{P}^1$ has two fixed points, and that intersection numbers are preserved by ι, that is, $D_1 \cdot D_2 = \iota^* D_1 \cdot \iota^* D_2$ for divisors D_i.

2.3. **I_n-fiber.** Let $\Theta_k \cong \mathbb{P}^1$ ($k \in \mathbb{Z}/n\mathbb{Z}$) be components of a fiber of type I_n ($n > 1$), such that Θ_0 intersects with the zero section o and that $\Theta_k \cdot \Theta_{k+1} = 1$ (or 2 if $n = 2$.). Then we can identify simple points of Θ_k with $\mathbb{C}^* \times \{k\} \subset \mathbb{C}^* \times \mathbb{Z}/n\mathbb{Z}$, replacing Θ_k by Θ_{-k} if necessary. There are two possibilities.

(i) If σ intersects with Θ_0 at the point corresponding to $(-1, 0) \in \mathbb{C}^* \times \mathbb{Z}/n\mathbb{Z}$, then ι acts on each Θ_k as an involution and fixed points are n intersection points $\Theta_k \cap \Theta_{k+1}$.

(ii) If $n = 2m$ and σ intersects with Θ_m at the point corresponding to $(\pm 1, m) \in \mathbb{C}^* \times \mathbb{Z}/2m\mathbb{Z}$, then ι switches Θ_k and Θ_{k+m} and there is no fixed point. In this case, we define a $\mathbb{Q}$-divisor

$$\vartheta_{2m} = \frac{1}{2m} \sum_{k=1}^{2m-1} k\Theta_k \in NS(X) \otimes \mathbb{Q}.$$

Note that $\vartheta_{2m} \cdot \Theta_k \in \mathbb{Z}$, $\vartheta_{2m} \cdot o = 0$, $\vartheta_{2m} \cdot \sigma = \frac{1}{2}$ and

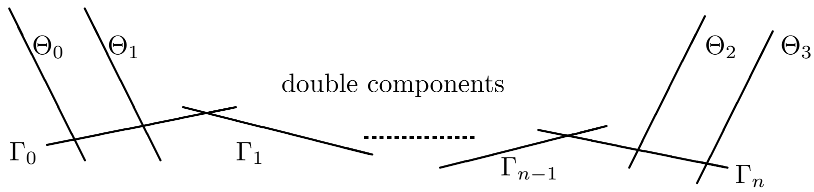
$$\begin{aligned} \vartheta_{2m} \cdot \vartheta_{2m} &= \frac{1}{4m^2}\{ \sum_{k=1}^{2m-1} k^2 \Theta_k \cdot \Theta_k + 2 \sum_{k=1}^{2m-2} k(k+1) \Theta_k \cdot \Theta_{k+1} \} \\ &= \frac{1}{4m^2}\{ -2 \sum_{k=1}^{2m-1} k^2 + 2 \sum_{k=1}^{2m-2} (k^2 + k) \} = -1 + \frac{1}{2m}. \end{aligned}$$

We shall use ϑ_{2m} later, to determine the discriminant group $NS(X)^*/NS(X)$.

2.4. **I_n^*-fiber.** Next, let $\Theta_0, \cdots, \Theta_3$ be simple components, and $\Gamma_1, \cdots, \Gamma_n$ be double components of a fiber of type I_n^* as in the following figure, and let Θ_0 be the component which intersects with o.

(i) If σ intersects with Θ_1, then ι switches Θ_0 and Θ_1, acts on each Γ_k and switches Θ_2 and Θ_3. In this case, we have n fixed points $\Gamma_k \cap \Gamma_{k+1}$ and another fixed point on Γ_1 and on Γ_n.

(ii) If $n = 2m$ and σ intersects with Θ_2 or Θ_3, then ι switches $\Theta_0 + \Theta_1$ and $\Theta_2 + \Theta_3$, acts on Γ_m and switches Γ_k and Γ_{2m-k}. In this case, we have 2 fixed points on Γ_m.



2.5. **Lemma.** Let X be an elliptic K3 surface with a 2-torsion and the Mordell-Weil rank 0.
(1) If the discriminant group T_X^*/T_X has a subgroup $\mathbb{Z}/p^e\mathbb{Z}$ for an odd prime number p, then X has a I_n-fiber with $n = kp^e$ for some $k \in \mathbb{N}$.
(2) If the discriminant group T_X^*/T_X has a subgroup $\mathbb{Z}/2^e\mathbb{Z}$ with $e \geq 3$, then X has a I_n-fiber with $n = 2^e k$ for some $k \in \mathbb{N}$.
(3) If the Picard number $\rho(X)$ is 17, then a singular fiber of X is one of the following:

$$I_1, \cdots, I_8, I_{10}, I_{12}, I_{14}, I_{16}, I_0^*, \cdots, I_6^*, I_8^*, I_{10}^*, III, III^*,$$

where $I_{10}, I_{12}, I_{14}, I_{16}, I_8^*$ and I_{10}^* are of type (ii). In particular, possible cyclic subgroups of T_X^*/T_X of order p^e are

$$\mathbb{Z}/2^e\mathbb{Z}\ (1 \leq e \leq 4), \quad \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/5\mathbb{Z}, \quad \mathbb{Z}/7\mathbb{Z}.$$

Proof. Let $L_X \subset NS(X)$ be the sublattic generated by the zero section o, a general fiber and components of singular fibers which do not intersect with o. Then L_X is of finite index in $NS(X)$, and we have

$$L_X \subset NS(X) \subset NS(X)^* \subset L_X^*.$$

Hence $T_X^*/T_X \cong NS(X)^*/NS(X)$ is isomorphic to a quotient of a subgroup of L_X^*/L_X. Note that

$$L_X \cong \mathrm{U} \bigoplus \Big(\bigoplus_{\Delta(\nu)=0} L_\nu \Big), \qquad L_X^*/L_X \cong \bigoplus_{\Delta(\nu)=0} L_\nu^*/L_\nu$$

and L_ν^*/L_ν is one of

$$\mathrm{A}_n^*/\mathrm{A}_n \cong \mathbb{Z}/n\mathbb{Z}, \quad \mathrm{D}_{2k}^*/\mathrm{D}_{2k} \cong (\mathbb{Z}/2\mathbb{Z})^2, \quad \mathrm{D}_{2k+1}^*/\mathrm{D}_{2k+1} \cong \mathbb{Z}/4\mathbb{Z}, \quad \mathrm{E}_7^*/\mathrm{E}_7 \cong \mathbb{Z}/2\mathbb{Z}$$

according to $I_n(III)$, I_{2k}^*, I_{2k+1}^* and III^*. Therefore subgroups $\mathbb{Z}/p^e\mathbb{Z}$ stated in (1) and (2) come from I_n-fibers.

By the Shioda-Tate formula ([Mi], Corollary VII.2.4)

$$\rho(X) = 2 + \mathrm{rank}\, X(K) + \sum_{\Delta(\nu)=0} (m_\nu(X) - 1),$$

and $\sum n_\nu = 8$, we see that a possible singular fiber is in the above list. □

2.6. **Theorem.** Let X be an elliptic K3 surface with a 2-torsion section σ which gives a Shioda-Inose structure. If $T_X \cong M_d$ and $\mathrm{rank} X(K) = 0$, then d is one of $1,\ 2,\ 3,\ 5,\ 7$ or 15. If $d = 15$, the singular fibers of X must be $6I_1 + I_2 + I_6 + I_{10}$ and the singular fibers of Y must be $6I_2 + I_4 + I_3 + I_5$. (As we shall see later, however, this configuration does not realize K3 surfaces with $T_X \cong M_{15}$.)

Proof. Since $T_X^*/T_X \cong \mathbb{Z}/2d\mathbb{Z}$, we see that a prime factor p of d is $2, 3, 5$ or 7, and that $p^2 \nmid d$ for $p = 3, 5, 7$. We have also $2^3 \nmid d$ since

$$T_Y^*/T_Y \cong M_d(2)^*/M_d(2) \cong (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4d\mathbb{Z}).$$

Now let q be the maximal prime factor of d.
(1) the case of $q = 7$. Let us show that $d = 7$. If $2|d$, then Y has $I_{7m} + I_{8n}$ and only $(m, n) = (1, 1)$ agrees with $\sum(m_\nu(Y) - 1) = 15$. However, $I_7 + I_8$ on Y corresponds to $I_{14} + I_4$ or $I_{14} + I_{16}$ on X, and both cases contradict $\sum(m_\nu(X) - 1) = 15$. Therefore 2 is not a prime factor of d. If $3|d$, then X has $I_{3m} + I_{7n}$ and only $(m, n) = (1, 1), (1, 2), (2, 1)$ agree with $\sum(m_\nu(X) - 1) = 15$. However, $I_3 + I_7$ has 10 fixed points by ι, and this contradicts $\sum n_\nu(X) = 8$. If X has $I_3 + I_{14}$, then singular fibers of X must be $I_3 + I_{14} + 7I_1$ by the conditions $\deg \Delta_X(t) = 24$ and $\sum(m_\nu(X) - 1) = 15$. This contradicts $\sum n_\nu(X) = 8$. We see that also $I_6 + I_7$ is impossible since it corresponds $I_{12} + I_{14}$ or $I_3 + I_{14}$ on Y. Therefore 3 is not a prime factor of d. By a similar argument, we can show that $5 \nmid d$.

(2) the case of $q = 5$. If $2|d$, then Y has $I_{5m} + I_{8n}$ and only $I_5 + I_8$ agrees with $\sum(m_\nu(Y) - 1) = 15$. This configuration is given as a degeneration (confluences of singular fibers)

$$8I_1 \ (b(t) = 0) \ + \ 8I_2 \ (c(t) = 0) \ \dashrightarrow \ (I_5 + 3I_1) + (I_8 + 4I_2).$$

of the most general configuration $8I_1 + 8I_2$. Under the hypothesis $\sum m_\nu(Y) = 15$, we may admit only $I_1 + I_2 \dashrightarrow III$ as extra confluences. By Corollary 1.7 in [Shio], we have

$$|\det NS(Y)| = \frac{\prod m_\nu^{(1)}(Y)}{|Y(K)_{tor}|^2} \le \frac{5 \cdot 8 \cdot 2^4}{4} < |\det M_{10}(2)|.$$

Therefore we see that 2 is not a prime factor of d, and we have $d = 5$ or $d = 15$. If $d = 15$, then X has one of

$$I_3 + I_5, \quad I_6 + I_{10}, \quad I_3 + I_{10}, \quad I_6 + I_5.$$

As degenerations of $8I_1 + 8I_2$, these are

$$(I_3 + I_5) + 8I_2, \quad 8I_1 + (I_6 + I_{10}), \quad (I_3 + I_5) + (I_6 + I_{10})$$

where $I_3 + I_{10}$ and $I_6 + I_5$ correspond to the same degeneration. From this, $\sum(m_\nu(X) - 1) = 15$ and the equality

$$\det NS(Y) = -\det M_{15}(2) = -2^5 \det M_{15} = 2^5 \det NS(X),$$

we see that the singular fibers of X and Y must be the stated form.

X	$8I_2 + 8I_1$	$\dashrightarrow$	$(I_6 + I_{10}) + 8I_1$	$\dashrightarrow$	$(I_6 + I_{10}) + (I_2 + 6I_1)$
Y	$8I_1 + 8I_2$	$\dashrightarrow$	$(I_3 + I_5) + 8I_2$	$\dashrightarrow$	$(I_3 + I_5) + (I_4 + 6I_2)$

(3) the case of $q = 3$. If $2|q$, then Y has $I_3 + I_8$ or $I_6 + I_8$, that is, the singular fibers of Y are obtained as a degeneration of one of the following two configurations.

X	$8I_2 + 8I_1$	$\dashrightarrow$	$(I_6 + 5I_2) + (I_4 + 4I_1)$ or $8I_2 + (I_3 + I_4 + I_1)$
Y	$8I_1 + 8I_2$	$\dashrightarrow$	$(I_3 + 5I_1) + (I_8 + 4I_2)$ or $8I_1 + (I_6 + I_8 + I_2)$

By the condition $\sum(m_\nu(Y) - 1) = 15$, we may admit just one of the following confluences

$$4I_k \dashrightarrow 2I_{2k}, \quad 3I_k \dashrightarrow I_{3k} \quad (k = 1, 2), \quad 2(I_1 + I_2) \dashrightarrow I_0^*,$$

and $I_1 + I_2 \dashrightarrow III$ if possible. In any cases, we have

$$\frac{\prod m_\nu^{(1)}(Y)}{|Y(K)_{tor}|^2} = |\det NS(Y)| = |\det T_Y|$$

$$= 2^5 |\det T_X| = 2^5 |\det NS(X)| = 2^5 \frac{\prod m_\nu^{(1)}(X)}{|X(K)_{tor}|^2}$$

and $|X(K)_{tor}| = 2^\varepsilon |Y(K)_{tor}|$ with $\varepsilon = 1, 0, -1$. Therefore we have an inequality

$$\prod m_\nu^{(1)}(Y) \le 2^3 \prod m_\nu^{(1)}(X)$$

and we see easily that this contradicts any case of the considering degenerations.
(4) the case of $q = 2$. If $4|d$, then Y must have I_{16}. In this case, the singular fibers of Y are $I_{16} + 8I_1$ and the singular fibers of X are $I_8 + 8I_2$. These K3 surfaces are studied in [vGS], and we have $T_X \cong M_2$ and $T_Y \cong M_2(2)$. □

3. Examples

3.1. For a cubic polynomial $P(t)$ and $0 \le n \le 8$, we define an elliptic K3 surface $X_d = X(d, P)$ by

$$y^2 = x(x^2 + P(t)x + t^d).$$

Then the quotient surface $Y_d = X_d / \langle \iota \rangle$ is

$$y^2 = x(x^2 - 2P(t)x + P(t)^2 - 4t^d).$$

The singular fibers of X_d and Y_d for a general $P(t)$ are given in the following table

	X_0	$X_d (1 \le d \le 6)$	X_7	X_8	Y_0	$Y_d (1 \le d \le 6)$	Y_7	Y_8
$t = 0$	reg.	I_{2d}	I_{14}	I_{16}	reg.	I_d	I_7	I_8
$c(t) = 0$	$6I_1$	$6I_1$	$7I_1$	$8I_1$	$6I_2$	$6I_2$	$7I_2$	$8I_2$
$t = \infty$	I_{12}^*	I_{12-2d}^*	III	reg.	I_6^*	I_{6-d}^*	III	reg.

where $c(t) = P(t)^2 - 4t^d$. Elliptic K3 surfaces X_1 were studied by Kumar in [K]. The transcendental lattice of a general X_1 is M_1 and the quotient surfaces Y_1 are Jacobian Kummer surfaces. Elliptic K3 surfaces X_8 were studied by van Geemen and Sarti in [vGS]. The transcendental lattice of a general X_8 is M_2 and the quotient surfaces Y_8 have the transcendental lattice $M_2(2)$.

3.2. **Proposition.** For a general cubic polynomial $P(t)$, we have
(1) the Picard number $\rho(X_d)$ is 17 for $d = 1, \cdots, 6$, and $\rho(X_0) = 18$,
(2) $X_d(K) = \{o,\ \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$ for $d = 1, \cdots, 6$,
(3) $\det NS(X_d) = 2d$ for $d = 1, \cdots, 6$, and $\det NS(X_0) = -1$. Hence $NS(X_0) \cong \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \mathrm{U}$ and $T_{X_0} \cong \mathrm{U} \oplus \mathrm{U}$.
(4) $T_{X_d} \cong M_d$ for $d = 1, \cdots, 6$.

Proof. (1) Cubic polynomials $P(t)$ form a 4-dimensional vector space, and we have isomorphisms

$$X(d, \lambda^{-4d}P(\lambda^8 t)) \longrightarrow X(d, P(t)), \quad (x, y, t) \mapsto (\lambda^{4d}x, \lambda^{6d}y, \lambda^8 t)$$

by $\lambda \in \mathbb{C}^*$. Up to this $\mathbb{C}^*$-action, the configuration of singular fibers is determined by $P(t)$ and it gives the moduli of $X(d, P)$ for $d = 1, \cdots, 7$. Therefore K3 surfaces $X(d, P)$ form a 3-dimensional family in this case. For $d = 0$, we can transform $P(t)$ into $t^3 + at + b$ by a transformation $t \mapsto \alpha t + \beta$, and (a, b) gives the moduli. From this, we see that $\rho(X_d) \le 17$ for $d = 1, \cdots, 7$ and $\rho(X_0) \le 18$. On the other hand, by the formula

$$\rho(X_d) = 2 + \operatorname{rank} X_d(K) + \sum_{\Delta(\nu)=0} (m_\nu(X_d) - 1),$$

we have

$$\rho(X_d) = \begin{cases} 18 + \operatorname{rank} X_0(K) & (d = 0) \\ 17 + \operatorname{rank} X_d(K) & (d = 1, \cdots, 6) \\ 16 + \operatorname{rank} X_7(K) & (d = 7). \end{cases}$$

Therefore we see that $\operatorname{rank} X_d(K) = 0$ for $d = 0, \cdots, 6$.
(2) We have an injective homomorphism $X_d(K)_{tor} \to (\mathbb{Z}/2\mathbb{Z})^2$ since $f^{-1}(\infty)^\sharp \cong \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2$. The 2-torsion subgroup of X_d is given by $o,\ \sigma$ and two solutions of $F(x) = x^2 + P(t)x + t^d = 0$. Since $F(x)$ is irreducible over K, we have

$$X_d(K) = X_d(K)_{tor} \cong \mathbb{Z}/2\mathbb{Z}$$

for $d = 0, \cdots, 6$.
(3) By Corollary 1.7 in [Shio], we have

$$|\det NS(X_d)| = \frac{\prod m_\nu^{(1)}(X_d)}{|X_d(K)_{tor}|^2} = \frac{1}{4}\prod m_\nu^{(1)}(X_d),$$

and $\det NS(X_d) = 2d$ for $d = 1, \cdots, 6$, and $\det NS(X_0) = -1$.
(4) The Néron-Severi group $N = NS(X_d)$ is generated by $o,\ \sigma$ and all components of singular fibers. Since the singular fiber at $t = 0$ is an I_{2d}-fiber of type (ii), we can define $\vartheta_{2d} \in N \otimes \mathbb{Q}$ as in 2.3. Let $\Theta_0, \cdots, \Theta_3$ be simple components of I^*_{12-6d}-fiber at $t = \infty$ as in 2.4. This fiber is of type (ii), and σ intersects with either Θ_2 or Θ_3. Let us consider

$$\Gamma = \frac{1}{2}(\Theta_2 + \Theta_3) + \vartheta_{2d} \in N \otimes \mathbb{Q}.$$

Since the intersection numbers of Γ with $o,\ \sigma$ and components of singular fibers are integers, we see that $\Gamma \in N^*$. Moreover we have $2d\Gamma \in N$ and the value of the discriminant form $q_N : N^*/N \to \mathbb{Q}/2\mathbb{Z}$ for Γ is

$$\Gamma \cdot \Gamma = \frac{1}{4}\{(\Theta_2)^2 + (\Theta_3)^2\} + (\vartheta_{2d})^2 = -2 + \frac{1}{2d} \equiv \frac{1}{2d} \mod 2.$$

If $m\Gamma \in N$, then we have $(\Gamma, m\Gamma) \in N^* \times N$ and

$$\frac{m}{2d} \equiv \Gamma \cdot (m\Gamma) \equiv 0 \mod \mathbb{Z}.$$

Therefore Γ gives an element of order $2d$ in N^*/N, and we have $N^*/N \cong \mathbb{Z}/2d\mathbb{Z}$. By Corollary 1.13.3 in [N2], we see that $N \cong \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \langle 2d \rangle$ and $T_{X_d} \cong M_d$. □

3.3. **Lemma.** For a general cubic polynomial $P(t)$, we have

$$\det NS(Y_d) = \begin{cases} 2^4 & (d=0) \\ 2^6 \cdot d & (d=1,3,5) \\ 2^4 \cdot d & (d=2,4,6) \end{cases}.$$

Proof. Since $Y_d(K)$ is isogeneous to $X_d(K)$, we see that $Y_d(K) = Y_d(K)_{tor}$. For $d = 0,2,4,6$, the group structure at $t = \infty$ is $\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2$, and we have full two-torsions:

$$y^2 = x(x^2 - 2P(t)x + P(t)^2 - 4t^d) = x(x - P(t) + 2t^{d/2})(x - P(t) - 2t^{d/2}).$$

Therefore the Mordell-Weil group $Y_d(K)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ in this case. For $d = 1,3,5$, the group structure at $t = \infty$ is $\mathbb{C} \times (\mathbb{Z}/4\mathbb{Z})$, and we have $Y_d(K) = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. If $\sigma' \in Y_d(K)$ has order four, then $\hat{\phi}(\sigma') \in X_d(K) \cong \mathbb{Z}/2\mathbb{Z}$ has order two. However, the non-zero element of $X_d(K)$ is pulled back to a double section

$$y = 0,\ x^2 - 2P(t)x + P(t)^2 - 4t^d = 0$$

of Y_d by $\hat{\phi}$. Hence we see that $Y_d(K) \cong \mathbb{Z}/2\mathbb{Z}$. As in the case of X_d, the Lemma follows from Corollary 1.7 in [Shio]. □

3.4. **Proposition.** Let $P(t)$ be a general cubic polynomial.
(1) The rational map $\phi : X_d \dashrightarrow Y_d$ gives a Shioda-Inose structure for $d = 0,1,3,5$. In particular, Y_d is a Kummer surface with the transcendental lattice $\mathrm{U}(2) \oplus \mathrm{U}(2)$ for $d = 0$, and $M_d(2)$ for $n = 1,3,5$.
(2) The transcendental lattice of Y_d is $\mathrm{U}(2) \oplus \mathrm{U}(2) \oplus \langle -d \rangle$ for $d = 2,4,6$.

Proof. We have a natural map $\phi_* : T_{X_d} \to T_{Y_d}$ between transcendental lattices such that $\phi_* T_{X_d} \cong T_{X_d}(2)$ (see [SI] and [Mo]).
(1) By Lemma 3.3, we see that $\det T_{X_d}(2) = \det T_{Y_d}$. Therefore we have $T_{X_d}(2) \cong T_{Y_d}$.
(2) By Lemma 3.3 and the conditions

$$\phi_* T_{X_d} \subset T_{Y_d} \subset (T_{Y_d})^* \subset (\phi_* T_{X_d})^*, \qquad (\phi_* T_{X_d})^* / \phi_* T_{X_d} \cong (\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/4d\mathbb{Z}),$$

we see that $(T_{Y_d})^*/T_{Y_d}$ is isomorphic to one of groups

$$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4n\mathbb{Z}), \quad (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2d\mathbb{Z}), \quad (\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/d\mathbb{Z}).$$

Let us consider a sublattice L of $N = NS(Y_d)$ generated by the zero section, a general fiber and components of singular fibers which does not intersect with the zero section. Then we have

$$L \subset N \subset N^* \subset L^*, \qquad L^*/L = (\mathbb{Z}/2\mathbb{Z})^8 \times (\mathbb{Z}/d\mathbb{Z})$$

since Y_d $(d = 2,4,6)$ has singular fibers I_d, $6I_2$ and I^*_{6-d}. Hence N^*/N does not contain an element of order $2d$, nor does $(T_{Y_d})^*/T_{Y_d}$. From this, we see that

$$(T_{Y_d})^*/T_{Y_d} \cong (\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/d\mathbb{Z})$$

and $T_{Y_d} \cong \mathrm{U}(2) \oplus \mathrm{U}(2) \oplus \langle -d \rangle$. □

3.5. Let us consider a family of elliptic K3 surfaces

$$X'_n : y^2 = x(x^2 + P(t)x + t^n(t-1)^{8-n}), \quad P(t) = 2t^4 - (8-n)t^3 + a_1t^2 + a_2t + a_3$$

for $n = 5,7$. A general X'_n has singular fibers I_{2n}, I_{16-2n}, I_2 and $6I_1$ at $t = 0,1,\infty$ and $P(t)^2 - 4t^n(t-1)^{8-n} = 0$, respectively. A general $Y'_n = X'_7/\langle\iota\rangle$ has singular fibers I_n, I_{8-n}, I_4 and $6I_2$ at $t = 0,1,\infty$ and $P(t)^2 - 4t^n(t-1)^{8-n} = 0$, respectively.

3.6. **Proposition.** For a general $P(t)$, we have
(1) $T_{X'_7} \cong M_7$ and $T_{X'_5} \cong \mathrm{U} \oplus \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \oplus \langle -6 \rangle$,
(2) $T_{Y'_n} \cong T_{X'_n}(2)$ for $n = 5,7$.
Proof. (1) Since $\sum(m_\nu(X'_n) - 1) = 15$ and a_1, a_2, a_3 give the moduli parameters, we have $\rho(X'_n) = 17$ and $\mathrm{rank} X'_n(K) = 0$. Then we have

$$\det NS(X'_n) = \frac{\prod m^{(1)}_\nu(X'_n)}{|X'_n(K)_{tor}|^2} = \frac{2n(8-n)\cdot 2^2}{|X'_n(K)_{tor}|^2}.$$

Since $2n(8-n)$ is square-free for $n=5,7$, we see that $X'_n(K)=\{o,\sigma\}$ and

$$\det NS(X'_n)=2n(8-n)=\begin{cases}30 & (n=5)\\ 14 & (n=7)\end{cases}$$

Note that singular fibers at 0 and 1 are of type (ii) and we have $\vartheta_{2n},\ \vartheta_{16-2n}\in NS(X'_n)\otimes\mathbb{Q}$. Since the singular fiber at ∞ is of type (i), the component Θ_1 does not intersects with o and σ. Then $\Gamma=\vartheta_{2n}+\vartheta_{16-2n}+\frac{1}{2}\Theta_1$ belongs to $NS(X'_n)^*$ and

$$\begin{aligned}\Gamma\cdot\Gamma &= (\vartheta_{2n})^2+(\vartheta_{16-2n})^2+(\frac{1}{2}\Theta_1)^2\\ &= (-1+\frac{1}{2n})+(-1+\frac{1}{16-2n})+(-\frac{1}{2})=\begin{cases}-2-\frac{7}{30} & (n=5)\\ -2+\frac{1}{14} & (n=7)\end{cases}.\end{aligned}$$

From this, we see that $NS(X'_7)\cong \mathrm{E}_8\oplus\mathrm{E}_8\oplus\langle 14\rangle$ and $T_{X'_7}\cong M_7$. Let $e_1,\cdots,e_5$ be the basis of $M=\mathrm{U}\oplus\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix}\oplus\langle -6\rangle$. Then we have

$$\delta=\frac{1}{5}(2e_3+e_4)+\frac{1}{6}e_5\in M^*$$

and δ generates $M^*/M\cong\mathbb{Z}/30\mathbb{Z}$. since $\delta\cdot\delta=\frac{7}{30}$, we see that $T_{X'_5}\cong M$ by Corollary 1.13.3 in [N2].
(2) We see easily that $Y'_n(K)\cong\mathbb{Z}/2\mathbb{Z}$ and $\det NS(Y'_n)=2^5\det NS(X'_n)$. Hence we have $T_{Y'_n}\cong\phi_*T_{X'_n}\cong T_{X'_n}(2)$. $\square$

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